Boğaziçi University

Workshop on Truth and Paradox

Session 4A: Bradwardine, Fitch and the Knower Paradox

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Closure Principles

If one knows that if p then q then if one knows that p, one knows that q.

$$(K) \qquad (\forall p)(\forall q)[\mathbf{Kn}(p \to q) \to (\mathbf{Kn}(p) \to \mathbf{Kn}(q))]$$

One knows that p and q iff one knows that p and one knows that q.

$$(K') \qquad (\forall p)(\forall q)[\mathbf{Kn}(p \land q) \leftrightarrow (\mathbf{Kn}(p) \land \mathbf{Kn}(q))]$$

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K entails K'

(1)	$\mathbf{Kn}(p \land q \to p)$	Given
(2)	$\mathbf{Kn}(p \land q \to q)$	Given
(3)	$\mathbf{Kn}(p \land q) \to \mathbf{Kn}(p)$	(1) by (K)
(4)	$\mathbf{Kn}(p \land q) \to \mathbf{Kn}(q)$	(2) by (K)
(5)	$\mathbf{Kn}(p \land q) \to (\mathbf{Kn}(p) \land \mathbf{Kn}(q))$	(3), (4) by Adjunction
(6)	$\mathbf{Kn}(p \to (q \to p \land q))$	Given
(7)	$\mathbf{Kn}(p) \to \mathbf{Kn}(q \to p \land q))$	(6) by (K)
(8)	$\mathbf{Kn}(p) \to \mathbf{Kn}(q) \to \mathbf{Kn}(p \land q))$	(7) by (K)
(9)	$(\mathbf{Kn}(p) \land \mathbf{Kn}(q)) \to \mathbf{Kn}(p \land q)$	(8) by Importation
(10)	$\mathbf{Kn}(p \land q) \leftrightarrow (\mathbf{Kn}(p) \land \mathbf{Kn}(q))$	(5), (9) by ($\leftrightarrow$I)

Also, knowledge is factive:

$$(T) \qquad (\forall p)(\mathbf{Kn}(p) \to p)$$

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The Principle of Knowability

Suppose that every truth were knowable:

$$(PK) \qquad (\forall p)(p \to \Diamond \mathbf{Kn}(p))$$

and there were a true unknown proposition p. Then by (PK):

$$p \land \neg \mathbf{Kn}(p) \to \Diamond \mathbf{Kn}(p \land \neg \mathbf{Kn}(p)) \qquad (*)$$

But

$\mathbf{Kn}(p \land \neg \mathbf{Kn}(p))$	$\to$	$\mathbf{Kn}(p) \land \mathbf{Kn}(\neg \mathbf{Kn}(p))$	by (K')
	$\to$	$\mathbf{Kn}(p) \land \neg \mathbf{Kn}(p)$	by (T)
Contradiction. So		$\neg \mathbf{Kn}(p \land \neg \mathbf{Kn}(p))$	
whence by Necessitation		$\neg \Diamond \mathbf{Kn}(p \land \neg \mathbf{Kn}(p))$	(**)
From (*) and (**) we can infer		$\neg(p \land \neg \mathbf{Kn}(p))$.	

i.e., there can be no such proposition p which is both true and unknown—given (PK), that no truth is unknowable. Since there clearly are truths we don't know, (PK) must be given up—not all truths can be known. Since there clearly are unknown truths, (PK) must be rejected.

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The Anonymous Referee's Conclusion

In the words of the anonymous referee, Alonzo Church:

> *"For each agent who is not omniscient, there is a true proposition which that agent cannot know."*

- The referee's insight was this: if one knew a truth to be true and unknown, it wouldn't be an unknown truth.
- There is no diagonalisation in Fitch's example. But consider, 'This proposition is true and unknown'.
- If it were known, it would be true and unknown, so it can't be known. Does that mean it's true?
- That's a fallacy. It's false, not because it's unknown, but because it's known to be unknown.

Propositional Quantification

All these principles quantify over propositions:

$$(K) \qquad (\forall p)(\forall q)[\mathbf{Kn}(p \to q) \to (\mathbf{Kn}(p) \to \mathbf{Kn}(q))]$$

$$(PK) \qquad (\forall p)[p \to \Diamond\mathbf{Kn}(p)]$$

$$(P2) \qquad (\forall p, q)((p \to q) \to (\mathbf{Sig}(s, p) \to \mathbf{Sig}(s, q)))$$

$$(D1) \qquad \mathbf{Tr}(s) := \mathbf{Prop}(s) \wedge (\forall p)(\mathbf{Sig}(s, p) \to p)$$

"for all propositions p, if s signifies that p then p ..." what?
"... then p is true"?—but then $(D1)$ is circular.
Actually, then $(D1)$ would be ill-formed.

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Ramsey's comment

Frank Ramsey wrote, in 'Facts and Propositions':

> *"... if I say 'He is always right', I mean that the propositions he asserts are always true, and there does not seem to be any way of expressing this without using the word 'true'. But suppose we put it thus 'For all p, if he asserts p, p is true' ... We have in English to add 'is true' to give the sentence a verb, forgetting that 'p' already contains a (variable) verb. This may perhaps be made clearer by supposing for a moment that only one form of proposition is in question, say the relational form aRb; then 'He is always right' could be expressed by 'For all a, R, b, if he asserts aRb, then aRb', to which 'is true' would be an obviously superfluous addition."*

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Semantics is Impossible

- In 's signifies that p', what replaces 's' is a name; but what replaces 'p' is a proposition, and *not* a name.
- If all expressions were names, semantics would be impossible—an error which, Coffa says, dogged the "semantic tradition" from the *Tractatus* through to Carnap's *Logical Syntax*.
- Ryle dubbed the doctrine that every expression is a name, the 'Fido'-Fido theory of meaning
- Moral: 'that p' is not a name, nor is 'p'.

Carnap met Tarski in a café in Vienna, and asked him about his definition of truth:

> *"I assumed that he had in mind a syntactical definition of logical truth or provability. I was surprised when he said that he meant truth in the customary sense, including contingent factual truth. Since I was thinking only in terms of a syntactical metalanguage, I wondered how it was possible to state the truth-condition for a simple sentence like 'this table is black.' Tarski replied, 'This is simple: the sentence "this table is black" is true if and only if this table is black.'"*

(T-scheme) $\qquad s$ is true if and only if p

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The grammar of 'says that'

- Arthur Prior: 'knows that' is a "connecticate", a predicate at one end and a (unary) connective at the other.
- In '*a* knows that *p*', 'that' is a relative conjunction, conjoining the main clause, '*a* knows', to the subordinate clause, '*p*'.
- Similarly, 'says that' is a connecticate, conjoining the subject-term, '*s*', to the subordinate clause, '*p*'.
- How then should, e.g. '$\mathbf{Sig}(s,p) \to p$' be read? It gives the form of propositions which result from it by substituting, in this case, a name for '*s*' and a proposition for '*p*', e.g. 'if "This table is black" signifies that this table is black then this table is black'.
- Quantification is not substitutional, however, on pain of cardinality constraints. Quantification needs a semantics, such as Greg Restall's, where propositional variables range over sets of worlds, i.e. maps from worlds to truth-values.

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Epistemic Paradox: the Knower Paradox

κ: You do not know this proposition

Suppose κ is false. Then no one knows it, including you. So κ is true. So by *reductio*, κ is true. Hence you do not know κ; but since you've proved that κ is true, you do know κ. Contradiction.

- Note that the argument of $\mathbf{Kn}(p)$ is not a name (like *s* in $\mathbf{Tr}(s)$), but a proposition.
- So we must express the Knower Paradox as:

  You do not know that this proposition is true.

- Suppose *s* signifies that it is not known that *s* is true.
- What else does *s* signify?

Suppose	$\mathbf{Sig}(s, \neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s})) \wedge q)$	
If $\mathbf{Fa}(s)$ then	$\exists p(\mathbf{Sig}(s,p) \wedge \neg p)$	$(D2)$
i.e.	either $\neg q$ or $\neg\neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s}))$	$(T1)$
So if $\mathbf{Fa}(s) \wedge q$	then $\mathbf{Kn}(\mathbf{Tr}(\mathrm{s}))$	Importation
So if $\neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s})) \wedge q$	then $\mathbf{Tr}(s)$	Antilogism
But	$\mathbf{Sig}(s, \neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s})) \wedge q)$	
So	$\mathbf{Sig}(s, \mathbf{Tr}(s))$	$(P2)$
i.e.	$\mathbf{Sig}(s, \neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s})) \wedge \mathbf{Tr}(s))$	

- So κ also signifies that κ is true, i.e. κ signifies that it is an unknown truth.

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Bradwardine's Second Thesis

- Recall Bradwardine's definitions, postulates and second theorem:

  "First Definition (D1): A true proposition is an utterance signifying only as things are.
  Second Definition (D2): A false proposition is an utterance signifying other than things are.
  . . .
  First Postulate (P1): Every proposition is true or false
  Second Postulate (P2): Every proposition signifies or means everything which follows from it . . .
  . . .
  Second Theorem (T2): If some proposition signifies itself not to be true or itself to be false, it signifies itself to be true and is false."

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Bradwardine's Third Thesis

D3 to know a proposition is to know wholly so to be as is signified by it.

T3 if a proposition only signifies itself not to be known by someone, or if in addition it only signifies some thing or things known by him, then it signifies that he does not know that he does not know it.

Proof: first, suppose $\neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s}))$ is all *s* signifies. Then:

$\neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s}))$	$\to$	$\neg\mathbf{Kn}(\neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s})))$	$(D3)$
But $\mathbf{Sig}(s, \neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s})))$	so	$\mathbf{Sig}(s, \neg\mathbf{Kn}(\neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s}))))$	$(P2)$

Now suppose $\mathbf{Sig}(s, \neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s})) \wedge q)$ where $\mathbf{Kn}(q)$:

$\neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s})) \wedge q$	$\to$	$\neg\mathbf{Kn}(\neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s})) \wedge q)$	$(D3)$
	$\to$	$\neg(\mathbf{Kn}(\neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s})) \wedge \mathbf{Kn}(q)))$	(K')
	$\to$	$\neg\mathbf{Kn}(\neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s})) \vee \neg\mathbf{Kn}(q))$	$(P4)$
But	$((\neg\mathbf{Kn}(\neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s})) \vee \neg\mathbf{Kn}(q))) \wedge \mathbf{Kn}(q)) \to \neg\mathbf{Kn}(\neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s})))$		$(P5)$
and $\mathbf{Kn}(q)$:	So	$\neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s})) \wedge q \to \neg\mathbf{Kn}(\neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s})))$	
So		$\mathbf{Sig}(s, \neg\mathbf{Kn}(\neg\mathbf{Kn}(\mathbf{Tr}(\mathrm{s}))))$	$(P2)$

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Diagnosis of the Knower Paradox

Suppose $\mathbf{Sig}(\kappa, \neg\mathbf{Kn}(\mathbf{Tr}(\kappa)))$. I'll call the inference from (a proof of) ϕ to $\mathbf{Kn}(\phi)$, Gödel's Rule. Here is the paradoxical reasoning:

$$\begin{array}{lllll}
\mathbf{Kn}(\mathbf{Tr}(\kappa)) & \to \quad \mathbf{Tr}(\kappa) & \to & \forall p(\mathbf{Sig}(\kappa, p) \to p) & (D1) \\
 & & \to & \neg\mathbf{Kn}(\mathbf{Tr}(\kappa)) & (T1) \\
\text{So} & \text{by} \quad \textit{reductio} & & \neg\mathbf{Kn}(\mathbf{Tr}(\kappa)) & \\
 & \quad\text{so} & & \mathbf{Tr}(\kappa) & (\mathbf{Tr}\text{-in})\dagger \\
 & \quad\text{so} & & \mathbf{Kn}(\mathbf{Tr}(\kappa)) & (\text{Gödel's Rule}) \\
 & \quad\text{and} & & \neg\mathbf{Kn}(\mathbf{Tr}(\kappa)) & (\mathbf{Tr}\text{-out}) \text{ or Rep}
\end{array}$$

Step (†) is wrong: pick it up at line (4):

$$\begin{array}{llll}
 & \text{So} & \mathbf{Kn}(\neg\mathbf{Kn}(\mathbf{Tr}(\kappa))) & (\text{Gödel's Rule}) \\
 & \text{But} & \mathbf{Sig}(\kappa, \neg\mathbf{Kn}(\neg\mathbf{Kn}(\mathbf{Tr}(\kappa)))) & (T3) \\
\text{So} & \exists p(\mathbf{Sig}(\kappa, p) \wedge \neg p) & \text{whence} \quad \mathbf{Fa}(\kappa) & (D2)
\end{array}$$

So the Knower is false, and not true, since something it signifies (though not that it's not known) fails to obtain.

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- However, Bradwardine's "proof" of ($T3$) has a gap:
- Recall the proof structure of ($T2.1$):

  T2.1 Suppose $\mathbf{Sig}(s, \neg\mathbf{Tr}(s))$
  - (a) and nothing else: then $\mathbf{Sig}(s, \mathbf{Tr}(s))$
  - (b) now suppose $\mathbf{Sig}(s, \neg\mathbf{Tr}(s) \wedge q)$: then $\mathbf{Sig}(s, \mathbf{Tr}(s))$
- Bradwardine doesn't add this further step in the proof of ($T3$):

  T3 Suppose $\mathbf{Sig}(s, \neg\mathbf{Kn}(\mathbf{Tr}(s)))$
  - (a) and nothing else: then $\mathbf{Sig}(s, \neg\mathbf{Kn}(\neg\mathbf{Kn}(\mathbf{Tr}(s))))$
  - (b) now suppose $\mathbf{Sig}(s, \neg\mathbf{Kn}(\mathbf{Tr}(s)) \wedge q) \wedge \mathbf{Kn}(q)$: then $\mathbf{Sig}(s, \neg\mathbf{Kn}(\neg\mathbf{Kn}(\mathbf{Tr}(s))))$
- But, as we have seen, $\mathbf{Kn}(\neg\mathbf{Kn}(\mathbf{Tr}(s)))$, so $\neg\mathbf{Kn}(\neg\mathbf{Kn}(\neg\mathbf{Kn}(\mathbf{Tr}(s))))$. We need a third case:
  - (c) now suppose $\mathbf{Sig}(s, \neg\mathbf{Kn}(\mathbf{Tr}(s)) \wedge q) \wedge \neg\mathbf{Kn}(q)$: then ???

  This final step resists proof. The proof is incomplete.

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- Fitch's Paradox is not a paradox: clearly, if one knew a truth to be true and unknown, it wouldn't be an unknown truth
- But the Knower Paradox is a paradox: (κ) 'You don't know this proposition' appears to be both true and known
- Bradwardine extended his dianosis of the semantic paradoxes to the epistemic paradoxes, in his third main thesis ($T3$): if a proposition signifies that it's not known, and in addition only signifies things that are known, then it signifies that it's not known that it's not known
- If ($T3$) were correct, that would suffice to show that the Knower Paradox was simply false
- Moreover (κ'), 'This proposition is both true and unknown' is equivalent to the Knower, and so it too is false
- However, there is a lacuna in Bradwardine's proof. Can it be filled?

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